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A Grothendieck module with applications to Poincaré rationality

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ABSTRACT

In this paper, we define a Grothendieck module associated to a Noetherian ring A. This structure is designed to encode relations between A-modules which can be responsible for the relations among Betti numbers and therefore rationality of the Poincaré series. We will define the Grothendieck module, demonstrate that the condition of being torsion in the Grothendieck module implies rationality of the Poincaré series, and provide examples. The paper concludes with an example which demonstrates that the condition of being torsion in the Grothendieck module is strictly stronger than having rational Poincaré series.

1. Introduction

Let A be a local Noetherian ring with residue field $\mathbf{k}$, and let M be a finitely generated A-module. We denote the Poincaré series of M by $P_A(M) = \sum_{i=0}^{\infty} \dim_{\mathbf{k}} \operatorname{Tor}_i(M, \mathbf{k})t^i$. When the module in question is the field $\mathbf{k}$ as a module over A, we refer to $P_A(\mathbf{k})$ as the Poincaré series of A, and denote it simply $P(A)$. (We refer the reader to [1–3] for background information.)

One of the first questions one asks about the Poincaré series is "Is it rational?" This is equivalent to asking about a linear recurrence relation among the Betti numbers (the coefficients of the series) (see [4] for an introduction to generating functions). Such a numerical relation suggests a corresponding structural relation among the Tor modules. The purpose of this paper is to suggest a mechanism by which such structural relations can be detected and encoded. A simple example will provide motivation.

Let $A = \mathbf{k}[x]/(x^a)$ for a field $\mathbf{k}$ and a positive integer a—the ring of truncated polynomials. We will demonstrate rationality of the Poincaré series of this ring. Let $\mathfrak{m} = (x)$ and note that $0 \to \mathfrak{m} \to A \to \mathbf{k} \to 0$ is exact. This short exact sequence produces a long exact sequence in Tor:

$$\cdots \to \operatorname{Tor}_{i+1}(A, \mathbf{k}) \to \operatorname{Tor}_{i+1}(\mathbf{k}, \mathbf{k}) \to \operatorname{Tor}_i(\mathfrak{m}, \mathbf{k}) \to \operatorname{Tor}_i(A, \mathbf{k}) \to \cdots.$$

Since $\operatorname{Tor}_i(A, \mathbf{k}) = 0$ for all $i \geq 1$, we get that $\operatorname{Tor}_{i+1}(\mathbf{k}, \mathbf{k}) \cong \operatorname{Tor}_i(\mathfrak{m}, \mathbf{k})$. Likewise, the short exact sequence $0 \to (x^{a-1}) \to A \to \mathfrak{m} \to 0$, with the right-hand map given by multiplication by x, gives rise to a long exact sequence which yields $\operatorname{Tor}_{i+1}(\mathfrak{m}, \mathbf{k}) \cong \operatorname{Tor}_i((x^{a-1}), \mathbf{k})$ for $i \geq 1$. Since $(x^{a-1}) \cong \mathbf{k}$, combining these yields $\operatorname{Tor}_{i+2}(\mathbf{k}, \mathbf{k}) \cong \operatorname{Tor}_i(\mathbf{k}, \mathbf{k})$ for all $i \geq 1$. But these Tor modules compute the Betti numbers, b_i, for the ring A, and so we conclude that $b_{i+2} = b_i$ for all $i \geq 1$. That is, the Poincaré series for the ring of truncated polynomials is rational and can be written with denominator $1 - t^2$.

The Grothendieck module, defined below, is designed to capture not just the relations among the Betti numbers, but also the relations between the modules themselves. This is accomplished by forming a $\mathbb{Z}[t, \frac{1}{t}]$-module generated by A-modules modulo relations defined by short exact sequences. This approach was inspired, in part, by the definition of the Grothendieck group, hence the name.

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Definition 1. Fix a left Noetherian ring A. Define the *Grothendieck module associated to* A, denoted $\mathcal{G}_A$, to be F_A/I_A where F_A is the free $\mathbb{Z}[t, \frac{1}{t}]$-module generated by isomorphism classes of finitely generated left A-modules and I_A is the submodule of F_A generated by the following classes of relations:

(R1) $[M]-[M']$ for all M and M' such that there exist a projective module P and an exact sequence $0 \to P \to M \to M' \to 0$;
(R2) $[M]-[M']$ for all M and M' such that there exist a projective module P and an exact sequence $0 \to M \to M' \to P \to 0$;
(R3) $[M] - t[M']$ (or equivalently $\frac{1}{t}[M] - [M']$) for all M and M' such that there exist a projective module P and an exact sequence $0 \to M' \to P \to M \to 0$; and
(R4) $[M \oplus M'] - [M] - [M']$ for all M and M'.

There is some redundancy in the relations (in particular, (R2) can be derived from (R1) and (R4)). We include the redundancy in the definition for clarity.

The main goals for this paper are to:

1. show that the condition that a module is torsion in the Grothendieck module implies rationality of the module's associated Poincaré series;
2. provide several illustrative examples; and
3. demonstrate that the condition of being torsion is in fact stronger than the condition of having rational Poincaré series.

2. Torsion implies rationality

Goal 1 is to show that a module which is torsion in the Grothendieck module has rational Poincaré series. The first task is to formalize the connection between the relations that define $\mathcal{G}_A$ and the corresponding relations between the Poincaré series of the modules.

Denote by $\mathbb{Z}((t))$ the ring of formal Laurent series, that is, the ring of formal expressions of the form $\sum_{i=n}^{\infty} a_i t^i$ where n is a (possibly negative) integer and the $a_i \in \mathbb{Z}$.

Lemma 1. *Let A be a commutative local Noetherian ring with residue field $\mathbf{k}$. There is a unique $\mathbb{Z}[t, \frac{1}{t}]$-homomorphism $\Phi : \mathcal{G}_A \to \mathbb{Z}((t))/\mathbb{Z}[t, \frac{1}{t}]$ taking the isomorphism class $[M]$ to the coset $P_A(M) + \mathbb{Z}[t, \frac{1}{t}]$.*

Proof. With notation as in Definition 1, let $\Psi : F_A \to \mathbb{Z}((t))$ be the unique $\mathbb{Z}[t, \frac{1}{t}]$-homomorphism taking the isomorphism class $[M]$ to $P_A(M)$. To complete the proof, it suffices to show that the image, under Ψ, of each of the generators of the submodule I_A is in $\mathbb{Z}[t, \frac{1}{t}]$. For relation (R4), we note that $\Psi([M \oplus M'] - [M] - [M']) = 0$ by the additivity of Tor. For the other relations, we provide the proof for only (R3), as the proofs for (R1) and (R2) are similar to it.

Let M and M' be finitely generated A-modules such that $0 \to M' \to P \to M \to 0$ is exact for some projective module P. Passing to the long exact sequence in Tor, we have

$$\cdots \to \operatorname{Tor}_{i+1}(P, \mathbf{k}) \to \operatorname{Tor}_{i+1}(M, \mathbf{k}) \to \operatorname{Tor}_i(M', \mathbf{k}) \to \operatorname{Tor}_i(P, \mathbf{k}) \to \cdots.$$

Set $b_i = \dim_{\mathbf{k}} \operatorname{Tor}_i(M, \mathbf{k})$ and $b'_i = \dim_{\mathbf{k}} \operatorname{Tor}_i(M', \mathbf{k})$. Using the fact that $\operatorname{Tor}_i(P, \mathbf{k}) = 0$, and thus $\operatorname{Tor}_{i+1}(M, \mathbf{k}) \cong \operatorname{Tor}_i(M', \mathbf{k})$, for all $i \geq 1$, we have that $b'_i = b_{i+1}$ for all $i \geq 1$ and thus:

$$\begin{aligned}
\Psi([M] - t[M']) &= \Psi([M]) - t\Psi([M']) \\
&= \sum_{i=0}^{\infty} b_i t^i - \sum_{i=0}^{\infty} b'_i t^{i+1} \\
&= b_0 + b_1 t - b'_0 t + \sum_{i=2}^{\infty} b_i t^i - \sum_{i=1}^{\infty} b_{i+1} t^{i+1} \\
&= b_0 + b_1 t - b'_0 t + \sum_{i=2}^{\infty} b_i t^i - \sum_{j=2}^{\infty} b_j t^j \\
&= b_0 + b_1 t - b'_0 t \in \mathbb{Z}\left[t, \frac{1}{t}\right]. \quad \square
\end{aligned}$$

Now we conclude that a module which is torsion in $\mathcal{G}_A$ has rational Poincaré series:

Theorem 2. *Let A be a commutative local Noetherian ring and let M be a finitely generated A-module. The Poincaré series of M is rational if and only if there exists a non-zero $f(t) \in \mathbb{Z}[t, \frac{1}{t}]$ such that $\Phi(f(t)[M]) = 0$. In particular, if $[M]$ is torsion in $\mathcal{G}_A$, then its Poincaré series is rational.*

Proof. Assume first that M has rational Poincaré series, say $P_A(M) = \frac{g(t)}{f(t)}$ for non-zero polynomials $g(t)$ and $f(t)$. Then $\Psi(f(t)[M]) = f(t)P_A(M) = g(t)$ and thus $\Phi(f(t)[M]) = g(t) + \mathbb{Z}[t, \frac{1}{t}] = 0$.

Conversely, assume $\Phi\,(f(t)[M]) = 0$ for some $f(t) \in \mathbb{Z}[t, \frac{1}{t}]$. That is, $\Psi(f(t)[M]) = f(t)P_A(M) = g(t)$ for some $g(t) \in \mathbb{Z}[t, \frac{1}{t}]$. Therefore

$$P_A(M) = \frac{g(t)}{f(t)}.$$

Clearing the denominators of f and g (they may have terms of negative degree) yields an expression for the Poincaré series as a quotient of polynomials. □

We have shown that the modules M with rational Poincaré series are exactly those for which there is a non-zero $f(t) \in \mathbb{Z}[t, \frac{1}{t}]$ such that $f(t)[M] \in \ker \Phi$. Our primary interest is in the trivial part of the kernel—when $f(t)[M] = 0$ in $\mathcal{G}_A$ for a non-zero $f(t)$. Goal 3 is the statement that the kernel is not always trivial.

3. Examples

This section provides two examples which make use of some classical results to obtain information about the Grothendieck module for certain classes of rings.

3.1. Regular rings

The Auslander–Buchsbaum–Serre theorem states that a local ring is regular if and only if the global dimension is finite (see [5]), and consequently the Poincaré series is polynomial. That is, regular local rings have the simplest possible Poincaré series. We will show that they also have the simplest possible Grothendieck modules, namely, the trivial module.

Proposition 3. *Let A be a commutative local Noetherian ring and M a finitely generated A-module. The projective dimension of M is finite if and only if $[M] = 0$ in $\mathcal{G}_A$.*

Proof. Assume that $\mathrm{pd}(M) = n < \infty$. Then there is a finite projective resolution of M of length n:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

By inserting kernels, we can break this exact sequence into short exact sequences:

$$\begin{aligned}
&0 \to P_n \to P_{n-1} \to \ker d_{n-1} \to 0\\
&0 \to \ker d_{n-1} \to P_{n-2} \to \ker d_{n-2} \to 0\\
&\vdots\\
&0 \to \ker d_1 \to P_1 \to \ker d_0 \to 0\\
&0 \to \ker d_0 \to P_0 \to M \to 0,
\end{aligned}$$

where $d_0 : P_0 \to M$ and $d_i : P_i \to P_{i-1}$ are the maps in the original sequence. By relation (R3), we see that: $0 = [P_n] = t^{-1}[\ker d_{n-1}] = t^{-2}[\ker d_{n-2}] = \cdots = t^{-(n-1)}[\ker d_1] = t^{-n}[M]$; so $t^{-n}[M] = 0$ and thus $[M] = 0$.

For the converse, assume that $[M] = 0$ in $\mathcal{G}_A$. Then $\Phi([M]) = 0$ in $\mathbb{Z}((t))/\mathbb{Z}[t, \frac{1}{t}]$. Since $\Phi([M])$ is the residue class of $P_A(M)$ in $\mathbb{Z}((t))/\mathbb{Z}[t, \frac{1}{t}]$, this means that $P_A(M)$ is a polynomial and the Betti numbers are eventually 0. Thus the minimal free resolution of M is finite, which implies finiteness of the projective dimension. □

Recall that a Noetherian ring is said to be regular if all localizations at prime ideals are regular local rings. Since localization is exact, the defining relations of $\mathcal{G}_A$ are preserved under localization. Also recall that any finitely generated module over the localization of a ring is the localization of a finitely generated module over the ring. Thus, $\mathcal{G}_A = 0$ implies that $\mathcal{G}_{A_p} = 0$ for all prime ideals p. Hence we have the following analogue to the Auslander–Buchsbaum–Serre theorem.

Corollary 4. *Let A be a Noetherian ring. Then $\mathcal{G}_A = 0$ if and only if A is regular.*

3.2. Hypersurfaces

It is known that modules over hypersurfaces have eventually periodic resolutions. In this section, we show that the Grothendieck module of a hypersurface is a torsion module. We begin by showing that any module with a periodic resolution is torsion.

Lemma 5. *Let A be a left Noetherian ring and M a finitely generated left A-module. Suppose that M has an eventually periodic resolution by finitely generated projective modules; that is, there is a resolution $\cdots \to P_i \xrightarrow{d_i} \cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$ such that, for some $n \geq 0$ and some $p \geq 1$, $P_{i+p} = P_i$ and $d_{i+p} = d_i$ for all $i \geq n$. Then $[M]$ is torsion in $\mathcal{G}_A$, and, in particular, $(1 - t^p)[M] = 0$.*

Proof. By the assumption that M is eventually periodic, we have the exact sequence

$$\cdots \to P_n \xrightarrow{d_n} P_{n+p-1} \to \cdots \to P_n \xrightarrow{d_n} \cdots.$$

Therefore both

$$0 \to \ker d_n \to P_n \to \cdots \to P_0 \to M \to 0$$

and

$$0 \to \ker d_n \to P_n \xrightarrow{d_n} P_{n+p-1} \to \cdots \to P_{n+1} \to \operatorname{im} d_{n+1} \to 0$$

are exact. Thus $\left(\frac{1}{t}\right)^{n+1}[M] = [\ker d_n]$, and, since $\operatorname{im} d_{n+1} = \ker d_n$, $t^p[\ker d_n] = [\ker d_n]$. Combining these, we see that $0 = (t^p - 1)[\ker d_n] = (t^p - 1)\left(\frac{1}{t}\right)^{n+1}[M]$. Therefore $(t^p - 1)[M] = 0$. □

Corollary 6. *Let A be a local hypersurface, that is, $A = R/(f)$ with $(R, \mathfrak{m}, \mathbf{k})$ a regular local ring and $f \in \mathfrak{m} - \mathfrak{m}^2$. Then $\mathcal{G}_A$ is a torsion module.*

Proof. For any finitely generated A-module M, Eisenbud [6] showed that the minimal resolution of M is eventually periodic with period 2. Hence, the previous result implies that $[M]$ is torsion for all finitely generated M, and so $\mathcal{G}_A$ is a torsion module. □

4. A class of rings for which [k] is not a torsion element

The final goal for this paper is to demonstrate that the property of being torsion in the Grothendieck module is stronger than the property of having a rational Poincaré series. The main result is:

Theorem 7. *Let $(A, \mathfrak{m}, \mathbf{k})$ be an Artinian local ring with A a complete intersection, that is, $A = B/(f_1, \ldots, f_c)$, where $(B, \mathfrak{n}, \mathbf{k})$ is a regular local ring of dimension c and $(f_1, \ldots, f_c)$ is a regular sequence in $\mathfrak{n}^2$. Then $P_A(\mathbf{k})$ is rational, but, for $c \geq 2$, $[\mathbf{k}]$ is not a torsion element in $\mathcal{G}_A$.*[1]

Before proceeding with the proof, the following preliminary result is necessary.

Lemma 8. *Let A be a local, Artinian, self-injective ring and let M be an indecomposable finitely generated module and M' a finitely generated module. Then $[M'] = [M]$ if and only if $M' \cong M \oplus A^k$ for some non-negative integer k.*

Proof. It is enough to show that the relations (R1) and (R2) amount to the existence of an isomorphism modulo a free module, and that $[M'] = t[N] = [M]$ and $[M'] = \frac{1}{t}[N] = [M]$ produce such isomorphisms as well.

For relation (R1): Assume that $0 \to P \to M' \to M \to 0$ is exact for a projective P. Since A is local, P is free, i.e., $P \cong A^k$ for some k. And since A is self-injective, the splitting property of injective modules implies that $M' \cong P \oplus M = A^k \oplus M$. Note that if the positions of M and M' are reversed, the assumption that M is indecomposable forces $k = 0$. Relation (R2) is similar.

Let N be a finitely generated module and assume that there exist projectives P and P' such that:

$$0 \to M \to P \to N \to 0$$
$$0 \to M' \to P' \to N \to 0.$$

That is, $[M] = \frac{1}{t}[N] = [M']$. Schanuel's lemma (see [1]) implies that $M \oplus P' \cong M' \oplus P$. Since A is local, P and P' are free and A is indecomposable as a module over itself. Moreover, since A is Artinian, decomposition into indecomposable modules is unique up to isomorphism and permutation. So, we have $M \oplus A^{k'} \cong M' \oplus A^k$ for some non-negative integers k and k', and every module is indecomposable with the possible exception of M'. So we can conclude $M' \cong M \oplus A^{k'-k}$.

The case $[M] = t[N] = [M']$ is similar, using self-injectivity and the analogous arguments for injective modules. □

In proving the main result of this section, it will be necessary to determine $t^n[\mathbf{k}]$ for all integers n. The lemma above means that identifying indecomposable modules S_n such that $[S_n] = t^n[\mathbf{k}]$ amounts to determining, up to isomorphism, all possible such modules.

Proof of Theorem 7. That $P_A(\mathbf{k})$ is rational was demonstrated by Tate (see [7]), among others. Assume that $c \geq 2$, that is, A is not a hypersurface. The goal is to show that $[\mathbf{k}]$ is not a torsion element of $\mathcal{G}_A$.

[1] I am deeply grateful to the anonymous referee who made significant improvements to the statement and proof of this result.

Begin with a minimal free resolution of $\mathbf{k}$:

$$\cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbf{k} \longrightarrow 0.$$

Since A is a complete intersection, it is Gorenstein, and consequently a complete resolution can be formed:

$$F_\bullet = \cdots \longrightarrow F_n \xrightarrow{\partial_n} F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \longrightarrow \cdots.$$

Letting $(\bullet)^* = \mathrm{Hom}_A(\bullet, A)$, we have $F_{-i} = F_{i-1}^*$ for all $i \geq 0$, $\partial_i = d_i$ and $\partial_{-i} = d_i^*$ for all $i \geq 1$, and $\partial_0 = d_0^* \circ d_0$ (see [8]).

For each integer i, set $S_i = \mathrm{im}(\partial_i)$. Then $S_0 = \mathrm{im}(\partial_i) = \mathrm{im}(d_0^* \circ d_0) \cong \mathbf{k}$, since the image of d_0 is $\mathbf{k}$ and d_0^* is injective. Clearly, then, $[S_i] = t^{-i}[\mathbf{k}]$, for each integer i.

The next claim is that each S_i is indecomposable. Then Lemma 8 will imply that the S_i uniquely, up to isomorphism and direct sum with free modules, identify the $t^{-i}[\mathbf{k}]$.

For $i \geq 0$, S_i is the ith syzygy of $\mathbf{k}$. Since A is Gorenstein and $\mathbf{k}$ is an indecomposable maximal Cohen–Macaulay module, S_i must be indecomposable (see [9]).

To see that S_{-i} is indecomposable for $i > 0$, consider the exact sequence

$$0 \to S_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to \mathbf{k} \to 0. \tag{†}$$

Dualize this sequence to obtain:

$$0 \to \mathbf{k}^* \to F_{-1} \to F_{-2} \to \cdots \to F_{-i} \to S_i^* \to 0.$$

Then $S_i^* \cong \mathrm{im}(d_i^*) = \mathrm{im}(\partial_{-i}) = S_{-i}$. If S_{-i} has a decomposition, say $S_{-i} \cong U \oplus V$, then $S_i \cong S_i^{**} = U^* \oplus V^*$. Thus either $U^* = 0$ or $V^* = 0$ and consequently $U = 0$ or $V = 0$.

Since each S_i is indecomposable and $[S_i] = t^{-i}[\mathbf{k}]$, Lemma 8 implies that if N is a finitely generated module such that $[N] = t^{-i}[\mathbf{k}]$, then $N \cong S_i \oplus A^k$ for some $k \geq 0$. The next step is to show that the S_i are distinct, that is, $S_i \ncong S_j$ for $i \neq j$.

First consider the case where $i > j \geq 0$. Since $F_\bullet$ began with a minimal resolution, the numbers of generators of S_i and S_j, for i and j non-negative, are equal to the ith and jth Betti numbers, respectively. Since A is assumed not to be a hypersurface, the Betti numbers are strictly increasing (see [10]) and thus $S_i \ncong S_j$.

Second, consider $0 \geq i > j$ and assume $S_i \cong S_j$. Then $S_i^* \cong S_j^*$, but, as was seen above, $S_i^* \cong S_{-i}$ and $S_j^* \cong S_{-j}$, yielding $S_{-i} \cong S_{-j}$, contradicting the case with both indices positive.

Finally, suppose that $S_i \cong S_j$ with $i > 0 > j$. Consider the exact sequence

$$0 \to S_i \to F_{i-1} \to \cdots \to F_0 \to F_{-1} \to \cdots \to F_j \to S_j \to 0.$$

Since $S_i \cong S_j$, copies of this exact sequence can be spliced onto Eq. (†) to obtain

$$\cdots \to S_i \to F_{i-1} \to \cdots \to F_j \to F_{i-1} \to \cdots \to F_1 \to F_0 \to \mathbf{k} \to 0.$$

This forms a periodic resolution of $\mathbf{k}$, which again contradicts the fact that the Betti numbers are strictly increasing. Therefore $S_i \ncong S_j$ and consequently $t^i[\mathbf{k}] \neq t^j[\mathbf{k}]$ for $i \neq j$.

Finally, assume that there exists some non-zero $f \in \mathbb{Z}[t, \frac{1}{t}]$ such that $f[\mathbf{k}] = 0$ in $\mathcal{G}_A$. Say $f = \sum_{i=-n}^m f_i t^i$. For each i, let $f_i^+ = f_i$ if $f_i > 0$ and 0 otherwise. Likewise, let $f_i^- = -f_i$ if $f_i < 0$ and 0 otherwise. Let $f^+ = \sum_{i=-n}^m f_i^+ t^i$ and let $f^- = \sum_{i=-n}^m f_i^- t^i$, so that $f = f^+ - f^-$. Note that f^+ and f^- have no terms in common and that $f[\mathbf{k}] = 0$ holds if and only if $f^+[\mathbf{k}] = f^-[\mathbf{k}]$. On the left-hand side, $f^+[\mathbf{k}] = \sum_{i=-n}^m f_i^+ t^i[\mathbf{k}] = [\oplus_{i=-n}^m S_{-i}^{f_i^+}]$, and likewise on the right. Again applying the lemma, we see that $\oplus_{i=-n}^m S_{-i}^{f_i^+} \oplus A^p \cong \oplus_{i=-n}^m S_{-i}^{f_i^-} \oplus A^q$. Since both sides of the isomorphism consist of indecomposable modules, we must have $f_i^+ = f_i^-$, that is, $f = 0$, a contradiction. Hence, $[\mathbf{k}]$ is not a torsion element in $\mathcal{G}_A$. □

We conclude that, for a local Noetherian ring A and finitely generated A-module M, if M is torsion in $\mathcal{G}_A$ then the Poincaré series of M is rational (this is Theorem 2); but the converse is false. That is to say, being a torsion element in the Grothendieck module is a stronger property than having rational Poincaré series.

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